Optimal Flight Paths for Soaring Flight

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The problem of optimizing sailplane flight paths to achieve maximum cross-country speeds with zero net altitude loss is considered. A variational formulation is chosen that includes three modes of cross-country soaring: thermalling, essing, and straight dolphining. Optimal solutions are obtained numerically for various atmospheric vertical velocity distributions using quadratic approximations to the polars of two sailplanes representing current high performance Standard and Open Class designs. Implications of the solutions are discussed; especially the optimality of any one mode when more than one mode is possible and the advance knowledge of the atmosphere required in order to choose an optimal speeds-to-fly policy. Particularly important is the result that the maximum cross-country speed through an element of the atmosphere capable of sustaining straight dolphin flight is attained with a speeds-to-fly policy identical to that of some equivalent interthermal flight.

Nomenclature

A,B,C	= coefficients of quadratic approximation to sailplane polar
h	= altitude relative to datum, ft
H	= Hamiltonian, min/ft
J	= cost function, min
ROC	= average rate of climb in a thermal, fpm
$(\mathrm{ROC})_{\mathrm{equiv}}$	$= -1/\lambda$, fpm
V_p	= downrange velocity, fpm
$V_p{}'$	= velocity at which $(V_{s,p})_{\min}$ occurs, fpm
$V_{x,a}$	= vertical atmospheric velocity (positive in the downward or sinking direction), fpm
$V_{s,a,\ell}$	= velocity of the rising segment of the atmosphere, fpm
$V_{s,a,s}$	= velocity of the sinking segment of the atmosphere, fpm
$V_{s,p}$	= sailplane characteristic sink rate, fpm
x	= downrange distance, ft
x_ℓ/x_f	= ratio of length of rising portion of atmospheric element to the total element length
λ	= Lagrange multiplier, min/ft
ϕ	= thermalling penalty term, min

Introduction

THERE has been a great increase of interest in recent years in problems associated with the optimization of sailplane flight paths to achieve maximum cross-country speeds. This interest has been stimulated by the evolution of competitive soaring into an almost exclusively cross-country racing sport and the concurrent development of high-performance sailplanes specifically designed for cross-country soaring.

Most cross-country flights are accompanied by gaining altitude by circling (thermaling) in localized areas with strong updrafts (thermals). The altitude thus gained is expended in straight line flight between thermals along the desired course heading. A simple algebraic expression can be derived for the average cross-country speed for this case, and a graphical method of obtaining the optimal speeds-to-fly between thermals is well known.[1]

If the lifting regions of the atmosphere are not extremely localized, but rather occur over some significant percentage of the course, it may be possible to maintain zero net altitude loss in straight flight along the desired course heading without circling. The resulting series of climbs and dives has led to the use of the term "dolphin soaring" to describe this mode of cross-country flying. In this paper it will be referred to as "straight dolphining" to distinguish it from the "thermaling" mode (which also exhibits dolphin motion between thermals, albeit with a net altitude loss) and from the "essing" mode.

Essing maneuvers (alternate right- and left-hand turns, not complete circles) performed in the lifting regions allow the pilot to extend his time in these regions and thus maintain zero net altitude loss without circling under less favorable atmospheric conditions than those necessary for straight dolphin flight.

It is generally recognized that many of the very fast cross-country flights achieved in recent years have been made under conditions where the latter two modes were utilized and relatively little time was spent in thermaling. Unfortunately, the optimal speeds-to-fly for these modes cannot be obtained as easily as those for the thermaling mode, and most previous printed discussion appears to be largely qualitative.[1,2] Arho[3] recently obtained the optimal speeds for a particular case of straight dolphin soaring with a sinusoidal atmospheric vertical velocity distribution, utilizing the calculus of variations. However, although the solution is correct, restriction to a single atmospheric distribution prevents any insight into the effect of variations in this distribution on the optimal speeds-to-fly. Irving[4] derives pairs of optimal speeds for straight dolphin flight through a uniform lifting region, followed by a uniform motionless section of the atmosphere. Questions about the optimality of the straight dolphin mode vs the thermaling or essing modes are left unanswered.

The present treatment attempts to provide at least some answers to these questions by treating all three modes of cross-country flying in the same variational problem formulation, and numerically determining the optimal speeds-to-fly for a variety of atmospheric vertical velocity distributions.

Problem Formulation

The present treatment is formulated in terms of a typical element of the atmosphere along the desired cross-country course. The horizontal component of the atmospheric velocity (wind) is assumed to be constant with respect to both altitude and downrange direction. The problem of wind distributions that are functions of downrange distance and/or altitude is not considered here. The vertical component has some

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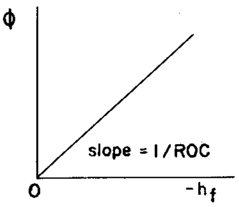

Fig. 1 Form of penalty function.

distribution $V_{s,a}$ (x) over the element, taken as positive in the downward direction. The object is to minimize flight time through the element with the restriction that the atltitude at the end of the element at x_f is the same as the altitude at the start $x=0$. If the $V_{s,a}$ distribution is such that this zero net altitude loss requirement cannot be satisfied with straight dolphining or essing, then the flight must include thermaling at the end of the element; the thermaling time will be included in the total flight time.

If the assumptions are made that a) the velocity of the sailplane can be controlled directly by the pilot (negligible elevator response time) and b) the sailplane horizontal velocity through the airmass is approximately equal to its airspeed, then the dynamic equations are

$$\dot{x}=V_p \tag{1}$$

$$\dot{h}=-[V_{s,a}+V_{s,p}] \tag{2}$$

where $V_{s,p}$ is the sailplane characteristic sink rate through the atmosphere as a function of airspeed, V_p. The order of the system can be reduced by changing the independent variable from time to downrange distance

$$\mathrm{d}h/\mathrm{d}x=-[V_{s,a}+V_{s,p}]/V_p \tag{3}$$

where $h(0)=0$ is chosen as the altitude datum for convenience.

The cost function for the problem is taken to be the time required to traverse the element range

$$j=\phi(h_f)+\int_0^{xf}\mathrm{d}x/V_p \tag{4}$$

where the integral term is the glide time from $x=0$ to $x=x_f$ and the penalty term is added if the final altitude at the end of the glide, h_f, is less than zero. This penalty term is simply the required thermaling, time $|h_f|/\mathrm{ROC}$, with ROC representing the average rate of climb in a thermal at x_f. Thus $\phi(h_f)$ has the form illustrated in Fig. 1.

The system Hamiltonian and associated Euler-Lagrange equations are[4]

$$H=(1/V_p)-\lambda[V_{s,a}+V_{s,p}]/V_p \tag{5}$$

$$\mathrm{d}h/\mathrm{d}x=-[V_{s,a}+V_{s,p}]/V_p \quad h(0)=0 \tag{6}$$

$$\mathrm{d}\lambda/\mathrm{d}x=-\partial H/\partial h=0 \quad \lambda(x_f)=\partial\phi(h_f)/\partial h_f \tag{7}$$

The optimal speeds-to-fly policy can be found using the well-known Minimum Principle[5]

$$V_{p,\mathrm{opt}}=\min_{V_p} H(V_p,\lambda,x) \tag{8}$$

This formulation is typical of that used in other flight path optimization solutions.[6] Equation (7) implies that $\lambda\equiv$ constant $=\partial\phi(h_f)/\partial h_f$; also, $\lambda\geq 0$ implies from Eq. (8) that $V_{p,\mathrm{opt}}=V_{p,\max}$ (red line) everywhere in the element with $h_f=0$. Although this situation is possible in regions of extremely strong atmospheric lift, such a $V_{s,a}$ distribution will not represent a typical element of any extended cross-country flight. Thus an adequate range of possible λ values is

$$h_f<0, \quad \lambda=-1/\mathrm{ROC} \tag{9a}$$

$$h_f=0, \quad -1/\mathrm{ROC}\leq\lambda<0 \tag{9b}$$

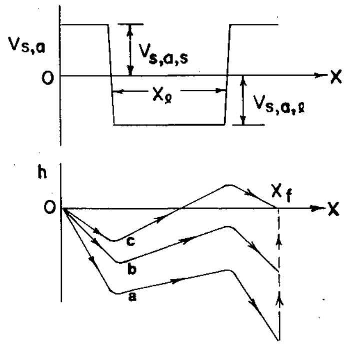

Fig. 2 Typical $V_{s,a}$ distribution and optimal flight paths.

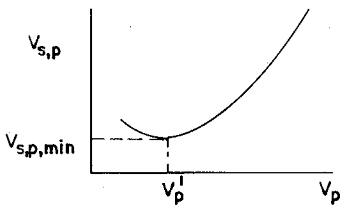

Fig. 3 Typical sailplane polar with essing modification.

For a given $V_{s,a}$ (x) distribution, the optimization procedure [Eqs. (5-8)] defines a set of optimal flight paths for a range of values of ROC down to some value $\mathrm{ROC}_{\mathrm{crit}}$. These paths all have $h_f<0$, and therefore include a thermal at x_f to return to the initial altitude. It should be noted that $h_f=0$ is not necessary to the optimization procedure; h_f could be any desired value. Also, it is not necessary that thermaling be done at x_f; the thermal could be located anywhere along the flight path.

Since the assumptions incorporated into the present formulation are identical with those implicit in the graphical speeds-to-fly determination, Eqs. (5-8) will prescribe optimal speeds identical with those of the graphical solution. (This is only true, however, if the graphical solution is utilized locally; i.e., a different flight speed for each different value of $V_{s,a}$. Flying at a constant speed between thermals based on an average value of $V_{s,a}$ between thermals is, in general, not optimal). A somewhat idealized $V_{s,a}$ distribution and typical optimal flight paths through it are sketched in Fig. 2. The set of thermaling solutions is illustrated by paths a and b for two different values of $\mathrm{ROC}>\mathrm{ROC}_{\mathrm{crit}}$.

Path c in Fig. 2, with $h_f=0$, describes the optimal flight path through the given $V_{s,a}$ distribution for all values of ROC $\leq\mathrm{ROC}_{\mathrm{crit}}$. In other words, the maximum cross-country speed can be realized in these cases by finishing the glide at $h_f=0$ and not utilizing the thermal. If the optimal speed-to-fly in the lifting region is greater than V_p at $V_{s,p,\min}$, then path c illustrates the straight dolphin mode and the optimization procedure is straightforward utilization of the normal sailplane polar ($V_{s,p}$ vs V_p).

However, if the optimum speed-to-fly in the lifting region is less than V_p at $V_{s,p,\min}$, then this indicates that essing is required in the region; and, since in this mode the sailplane x-component of horizontal velocity is no longer equal to its airspeed, the normal sailplane polar must be modified to reflect this fact. The approach taken in the present work utilizes an equivalent straightline segment for the essing region with $V_{s,p}=V_{s,p,\min}$ at all values of V_p down to zero, where V_p in this region is the net downrange velocity component. Figure 3 illustrates this equivalent polar.

The optimality of flight in the dashed region of Fig. 3 requires special consideration. For essing in the lifting region of a distribution like that of Fig. 2, $V_{s,a}=V_{s,a,\ell}=$ constant. So Eq. (5) becomes

$$H=\frac{[1-\lambda(V_{s,a,\ell}+V_{s,p,\min})]}{V_p}=\frac{\text{constant}}{V_p} \tag{10}$$

The minimum principle [Eq. (8)] yields

$$V_p = 0 \quad \text{for } 1-\lambda(V_{s,a,\ell}+V_{s,p,\min})<0 \tag{11a}$$

$$V_p = V_p' \quad \text{for } 1-\lambda(V_{s,a,\ell}+V_{s,p,\min})>0 \tag{11b}$$

However, a choice of λ such that $V_p=0$ or V_p' will not, in general, satisfy $h_f=0$. The proper choice of λ for the present situation must be one such that $1-\lambda(V_{s,a,\ell}+V_{s,p,\min})\equiv 0$. This choice gives rise to a singular arc $(\partial^2 H/\partial V_p^2=0)$ in the essing region. This allows V_p to be chosen in the essing region so that $h_f=0$ and still satisfy the necessary conditions as long as $0\leq V_p \leq V_p'$. It should be noted that the optimization does not define the details of the essing maneuver. Essing calls for constant airspeed at V_p', and the optimization defines only the effective ground speed the maneuver must achieve.

Numerical Results and Discussion

Numerical solutions to the optimization problems just posed have been obtained using quadratic approximations of the polars of two sailplanes, Standard Libelle and Nimbus II, which represent current high-performance competitive standard and open-class sailplanes, respectively. For both, $V_{s,p}$ vs V_p, with the essing modification, is given by

$$V_{s,p} = AV_p^2 + BV_p + C \qquad V_p \geq V_p' \tag{12a}$$

$$V_{s,p} = V_{s,p,\min} \qquad 0\leq V_p < V_p' \tag{12b}$$

For the Standard Libelle, $A=14.7\times10^{-6}$ min/ft, $B=-12.4\times10^{-2}$, $C=3.97\times10^{2}$ fpm, $V_p'=4.36\times10^{3}$ fpm, and $V_{s,p,\min}=134$ fpm with all velocities expressed in fpm. Values calculated with this approximation differ from the measured values presented by Bikle[7] by a maximum of 7%. For the Nimbus II, $A=9.48\times10^{-6}$ min/ft, $B=-7.78\times10^{-2}$, $C=2.54\times10^{2}$ fpm, $V_p'=4.10\times10^{3}$ fpm, and $V_{s,p,\min}=94.5$ fpm. This approximation is the same one as used by Arho.[3]

The optimal flight paths and speeds-to-fly were obtained for flight through a variety of $V_{s,a}$ distributions of the form given in Fig. 2. The ratio of the length of the lifting portion of the element to the total element length x_ℓ/x_f was varied for each of several combinations of $V_{s,a,s}$ and $V_{s,a,\ell}$. The optimal solution, as represented by the value of the Lagrange multiplier λ, was obtained by a simple iterative procedure with the range $x=0$ to $x=x_f$ divided into 50 equal steps: a) assume a value of λ between $-1/\text{ROC}$ and zero; b) at each x location, minimize Eq. (5) with respect to V_p; c) integrate Eq. (6) from $x=0$ to $x=x_f$ and d) check to see if Eq. (9) is satisfied; if not, repeat with a new value of λ. For example, for the case $V_{s,a,s}=500$ fpm, $V_{s,a,\ell}=-500$ fpm, $x_\ell/x_f=0.80$ with the Standard Libelle characteristic, the numerical procedure results in a value $\lambda=-1.12\times10^{-3}$ min/ft. The corresponding optimal speeds are 11,050 fpm (126 mph) in the sinking portion of the element and 7340 fpm (83 mph) in the lifting portion. There is no net altitude loss $(h_f=0)$ and, since the airspeed in the lift is greater than V_p', this is a case where the element can be traversed by straight dolphining.

The particular value of λ obtained has some important physical significance that can be recognized if it is noted that the negative reciprocal of λ (in this case 893 fpm) has the same function in the governing equations for the straight dolphining and essing modes as the expected rate of climb does in the thermaling mode. Thus, the airspeeds given are identical with the normal interthermal optimal speeds through this $V_{s,a}$ distribution with an expected ROC of 893 fpm in a thermal. It also follows that the optimal flight path through this particular $V_{s,a}$ distribution does not include thermaling unless a rate of climb in a thermal greater than 893 fpm can be achieved.

With this interpretation of λ in mind, the optimal solutions are presented as $-1/\lambda=(\text{ROC})_{\text{equiv}}$, as in Fig. 4 for other values of x_ℓ/x_f with $V_{s,a,s}=500$ fpm and $V_{s,a,\ell}=-500$ fpm.

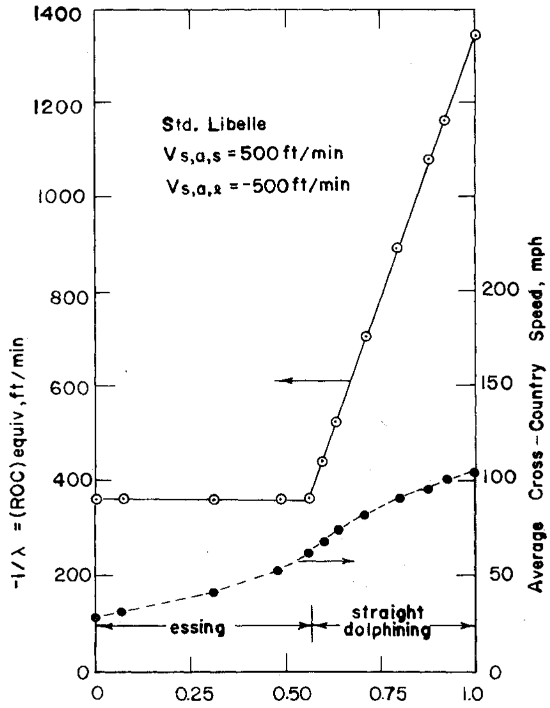

Fig. 4 Typical variation of $(\text{ROC})_{\text{equiv.}}$ and average cross-country speed with x_ℓ/x_f.

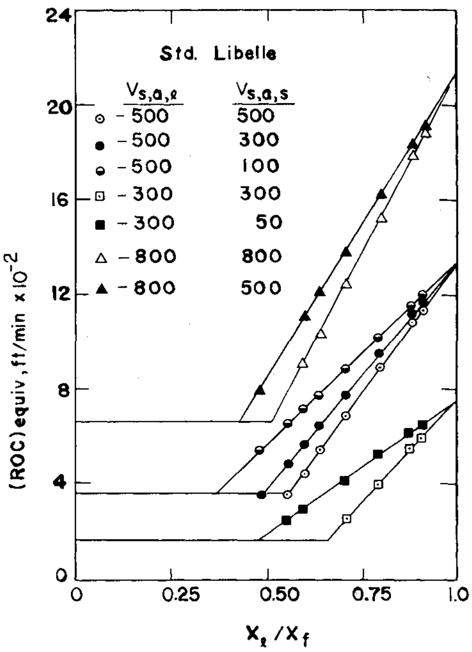

Fig. 5 $(\text{ROC})_{\text{equiv.}}$ vs x_ℓ/x_f for the standard Libelle.

As x_ℓ/x_f increases toward 100% of the element, $(\text{ROC})_{\text{equiv}}$ increases and the optimal speed-to-fly in the lift increases toward the value (105 mph) which gives $V_{s,p}=500$ fpm. Conversely, as x_ℓ/x_f decreases, $(\text{ROC})_{\text{equiv}}$ decreases until a value of 366 fpm is reached at $x_\ell/x_f=0.56$. At this point the optimal speed-to-fly in the lift is exactly equal to V_p'. Any further reduction of x_ℓ/x_f requires essing in the lifting region to satisfy $h_f=0$. The optimal speeds-to-fly for this and all smaller values of x_ℓ/x_f correspond to $(\text{ROC})_{\text{equiv}}$ of 366 fpm, which is the maximum with this particular sailplane $(V_{s,p,\min}=134$ fpm$)$ in 500 fpm lift. Also presented in Fig. 4 are the average cross-country speeds corresponding to the optimal flight paths.

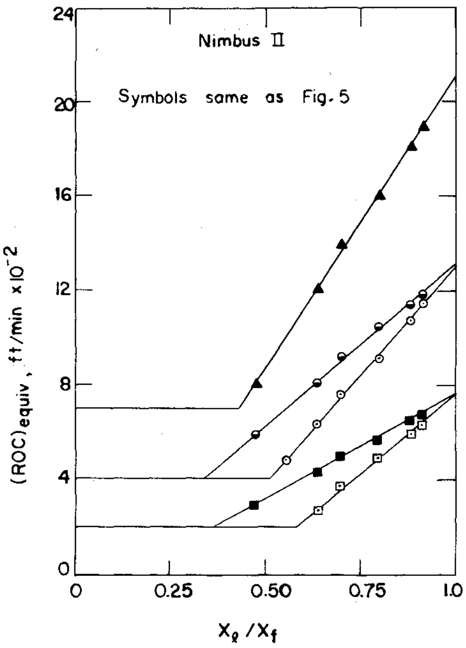

Fig. 6 $(ROC)_{equiv.}$ vs x_ℓ/x_f for the Nimbus II.

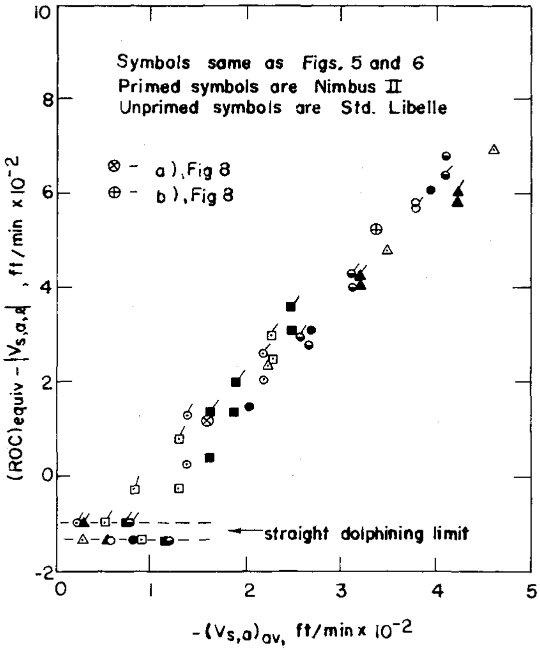

Fig. 7 Correlation of results in the straight dolphining region.

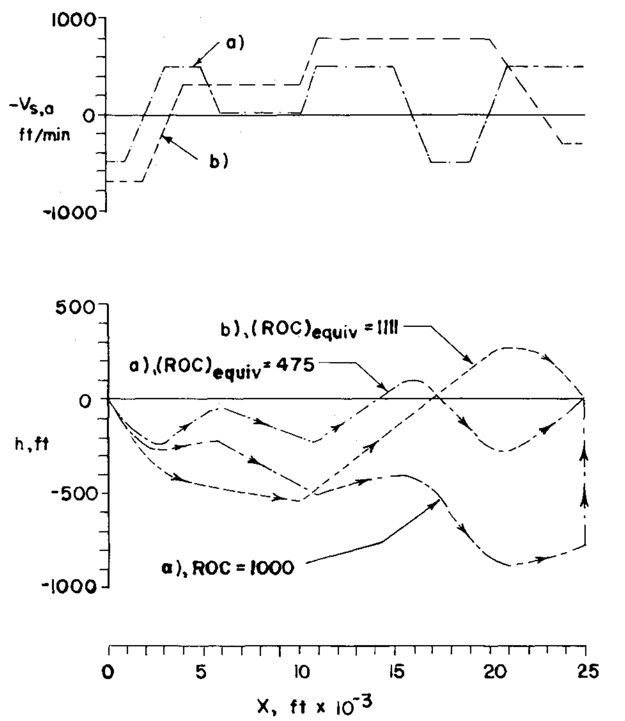

Fig. 8 $V_{s,a}$ distributions (a) and (b) and optimal flight paths with standard Libelle.

Figure 5 presents similar $(ROC)_{equiv}$ variations obtained for the Standard Libelle with other values of $V_{s,a,\ell}$ and $V_{s,a,s}$. All distributions with the same value of $V_{s,a,\ell}$ have the same value of $(ROC)_{equiv}$ in the essing region and at $x_\ell/x_f = 1$. The almost perfectly linear dependence of $(ROC)_{equiv}$ on x_ℓ/x_f in the straight dolphining region is exhibited through the spectrum of distributions investigated, and probably results from the quadratic form of the sailplane polar.

Figure 6 summarizes the corresponding cases evaluated with the Nimbus II characteristic. The values of $(ROC)_{equiv.}$ in the essing region are higher than those of Fig. 5 because of the smaller value of $V_{s,p,min}$ for the Nimbus. Also, for the lower values of $V_{s,a,\ell}$, the minimum values of x_ℓ/x_f necessary to sustain the straight dolphining mode are smaller for the Nimbus. Otherwise, these results for the two planes are quite similar. Of course, even where the $(ROC)_{equiv.}$ values are essentially identical for the two sailplanes, this does not imply identical performance. For example, at $V_{s,a,s} = 500$ fpm, $V_{s,a,\ell} = -500$ fpm, and $x_\ell/x_f = 0.80$, the optimal flight path for both planes is flown with an $(ROC)_{equiv.}$ value of approximately 900 fpm. The average cross-country speeds, however, are significantly different: 104 mph for the Nimbus as compared with 90 mph for the Libelle.

This example is indicative of the fact that very fast cross-country speeds can be achieved where straight dolphining can be utilized, and these speeds are attained by flying in accordance with much higher ROC values than could actually be achieved by thermaling or essing in the lifting portions of the flight. Unfortunately, to choose the optimal speeds, the character of the $V_{s,a}$ distribution must be known in advance. Even for the simple distributions investigated, three quantities ($V_{s,a,s}$, $V_{s,a,\ell}$, and x_ℓ/x_f) must be known in order to choose the proper $(ROC)_{equiv.}$. This is in contrast with the thermaling mode, where the required advance knowledge collapses to the expected average rate of climb in the next thermal. A logical question at this point is, whether there is some simpler description of the $V_{s,a}$ distribution possible that could be used in flight to predict in advance the proper $(ROC)_{equiv.}$ value.

The answer to this question is a tentative yes, at least for the cases presented in Figs. 5 and 6. Figure 7 shows the correlation of these results in the straight dolphining region that can be achieved by presenting $(ROC)_{equiv.} - |V_{s,a,\ell}|$ vs the spatial average of $V_{s,a}$ over the entire range. Thus, if the magnitude of the lift $|V_{s,a,\ell}|$ and the average vertical velocity of the atmosphere can be estimated, an estimate of the proper $(ROC)_{equiv.}$ value can be made. The lower left-hand points for each set of conditions in Fig. 7 correspond to the minimum value of x_ℓ/x_f that will sustain straight dolphin flight. It can be seen that this mode of flying is sometimes possible even when the average atmospheric lift is significantly less than the minimum sink rate of the sailplane.

To test the implications of Fig. 7 with more complex $V_{s,a}$ distributions, optimal flight paths for the distributions a and b of Fig. 8 have been obtained. These have $(V_{s,a})_{av}$ values of -160 and -338 fpm, respectively. Relative altitudes in feet are pictured for the case where $x_f = 25{,}000$ ft, using the Standard Libelle characteristics. Straight dolphin flight is possible through both of these distributions, as illustrated by the two paths which end at $h = 0$ at $x = x_f$. For a and b the $(ROC)_{equiv.}$ values are 475 fpm and 1111 fpm, respectively; and the corresponding average cross-country speeds are 72.1 and 95.5 mph. The spatial average of $V_{s,a}$ over regions where $V_{s,a} \leq 0$ for distributions a and b are roughly -350 and -650 fpm, respectively. If these are used for equivalent values of $V_{s,a,t}$ in Fig. 7, the $(\mathrm{ROC})_{\mathrm{equiv.}}$ results from these two distributions a and b are seen to fall within the band of previous values. This suggests that estimation of two quantities, $(V_{s,a})_{\mathrm{av}}$ over the entire region and $(V_{s,a})_{\mathrm{av}}$ over the lifting portion alone, may be quite adequate for making an estimate of the proper $(\mathrm{ROC})_{\mathrm{equiv.}}$ value when entering a region capable of sustaining straight dolphin flight. Presumably, as flight progresses through the region, these initial estimates can be updated. It should be remembered that only $(\mathrm{ROC})_{\mathrm{equiv.}}$ and the local value of $V_{s,a}$ are needed to find $V_{p,\mathrm{opt}}$.

Figure 8 also includes an optimal flight path for distribution a) when an average rate of climb of 1000 fpm can be achieved in a thermal at x_f. In this case, the average cross-country speed is 79.3 mph, compared with the 72.1 mph in the straight dolphining mode. This illustrates that maximum cross-country speeds can sometimes be achieved by circling in the strongest areas of lift, even though circling is not necessary for maintaining altitude.

The two optimal paths shown in Fig. 8 for distribution a) also graphically illustrate the presence of dolphin-type motion in both cases, reflecting the fact that they are both members of the same family of optimal solutions. Along these lines it is useful to note that all of the optimal solutions covered in the present treatment ($h_f=0$) are members of the much larger family of solutions where $h_f \neq 0$. There is nothing special about the choice of $h_f=0$ in the mathematical formulation; and, using Fig. 8 as an example, the problem could have been to find the optimal flight path through distribution a) to a final altitude of -785 ft. The optimal path for this problem is precisely that already sketched in Fig. 8, without the thermal at the end. $(\mathrm{ROC})_{\mathrm{equiv}}$ for this case is 1000 fpm, and if ROC >1000 fpm could be achieved in a thermal, then it should be utilized to reduce the flight time further.

Conclusions

The most important conclusion of the present study is that the various modes of cross-country soaring can be viewed in the context of a single variational problem. Maximum cross-country speeds are achieved with flight paths dictated by the characteristic values (Lagrange multipliers) of solutions to this problem. In the cases of thermaling or essing, the characteristic value is specified solely by the portion of the flight path where thermaling or essing takes place, and is simply the negative reciprocal of the average rate of climb achieved while thermaling or essing.

In the case of straight dolphining, the characteristic value depends on a more detailed description of the atmospheric lift-and-sink distribution, but can still be interpreted as an equivalent average rate of climb. Thus, the common graphical determination of best speeds-to-fly, using average rate of climb as a parameter, can, properly interpreted, be used for optimizing the straight dolphining and essing modes as well as for the thermaling mode.

The numerical results obtained in this study indicate that the equivalent average rates of climb in the straight dolphining region correlate together quite well for different sailplanes and different atmospheric vertical velocity distributions in terms of the spatial average of the distribution and the average in the lifting portion of the distribution alone. Thus, if a reasonable estimate of these two quantities can be made in advance (analogous to estimating the expected rate of climb in the next thermal), then the optimal straight dolphining speeds-to-fly can be chosen.

If straight dolphin flight without net altitude loss is not possible, and if the maximum rate of climb can be achieved over a large area (large with respect to the turning diameter of the plane) then maximum cross-country speeds can be attained by essing through the band of maximum lift. Conversely, if the maximum achievable rate of climb is localized, it is better to thermal at that location.

Admittedly, from the pilot's point of view, such rules are very often much easier to state than to follow in practice. Moreover, the present treatment does not include any of the constraints which usually affect cross-country flying tactics, such as altitude maximums dictated by cloud base, proximity to the ground (which strongly influences the pilot to fly at speeds which maximize his range, rather than his cross-country speed) etc. Nevertheless, familiarity with at least the general characteristics of a wider class of optimal solutions than that of the thermaling case would appear to be desirable.

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